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LINEAR INVARIANT MEASURES FOR RECURRENT LINEAR SYSTEMS

BY

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ABSTRACT

We consider a self-adjoint operator defined by a bidimensional linear system. We extend the Ishii–Pastur–Kotani theory that allows us to identify the absolutely continuous spectrum. From here we deduce that for almost every E with null Lyapunov exponent the real projective flow admits absolutely continuous invariant measures with square integrable density function.

1. Introduction

We introduce self-adjoint operators associated to a one-parameter family of bidimensional linear systems. Each linear equation induces a continuous flow in the projective bundle where we define a product measure in a natural way. This article connects two topics with each other, of spectral and ergodic nature repectively, which are apparently different.

We concentrate the attention on the real axis and deduce our conclusions from a same result: the limit with imaginary part of the Weyl–Titchmarsh functions. From here we derive the absolutely continuous part of the spectral measure. For adequate values of the parameter we obtain an invariant measure on the projective bundle which is absolutely continuous with respect to the product measure.

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In order to make this paper reasonably self-contained, we briefly describe some notation and results more or less standard in this theory.

Let us consider the real linear system

$$\mathbf{z}' = \begin{pmatrix} a_0(t) & b_0(t) \\ c_0(t) & -a_0(t) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = S_0(t)\mathbf{z} \tag{1}$$

whose matrix of coefficients is uniformly continuous, bounded and with trace zero.

Let Ω be the hull of S_0, that is $\Omega = \text{cls}\{S_{0,t}(s) = S_0(t+s) | \ t \in \mathbb{R}\}$ in the topology of the uniform convergence on the compact sets. Clearly $\Omega \subset C(\mathbb{R}, \text{sl}(2,\mathbb{R}))$ and we represent by $\xi \in \Omega$ any of its elements. The translation $\mathbb{R} \times \Omega \to \Omega$, $(t, \xi(s)) \to \xi(t+s)$ defines a flow Ξ on Ω. Let $S\colon \Omega \to \text{sl}(2,\mathbb{R})$ be the operator which takes ξ to $\xi(0)$. Taking $\xi = S_0$ we obtain $S(\xi_t) = S_0(t)$ which indicates that S_0 can be recuperated evaluating S along a trajectory. In this way S_0 is extended to a function $S \in C(\Omega, \text{sl}(2,\mathbb{R}))$ which leads us to consider the family of linear systems

$$\mathbf{z}' = \begin{pmatrix} a(\xi_t) & b(\xi_t) \\ c(\xi_t) & -a(\xi_t) \end{pmatrix} \mathbf{z} = S(\xi_t)\mathbf{z} \tag{2}$$

where (1) is included.

Let us suppose that the flow (Ω, Ξ) is minimal, then the system (1) is often called recurrent. We fix an ergodic measure m_0 for all that follows and will denote by $\mathcal{A}_0$ the complection of the σ-algebra of Borel sets with respect to m_0. The symbol r will stand for the Lebesgue measure on $\mathbb{R}$.

The equations (2) induce a skew-product flow in the linear bundle $V_{\mathbb{C}} = \Omega \times \mathbb{C}^2$ ($V_{\mathbb{R}} = \Omega \times \mathbb{R}^2$). It takes $(\xi, \mathbf{z}_0) \cdot t$ to $(\xi_t, \mathbf{z}_t(\xi, \mathbf{z}_0))$ where $\mathbf{z}_t(\xi, \mathbf{z}_0)$ verifies the equation defined by (2) along the trajectory that passes through ξ with the initial data $\mathbf{z}_0(\xi, \mathbf{z}_0) = \mathbf{z}_0$. By linearity on the fibres the application $\Pi\colon V_{\mathbb{C}} \to K_{\mathbb{C}} = \Omega \times P^1(\mathbb{C})$, $(\xi, \mathbf{z}) \to (\xi, z_2/z_1)$ transports this flow to the projective bundle. The symbol Φ will represent the flow application in any invariant subset of $K_{\mathbb{C}}$ that is considered.

As usual, $P^1(\mathbb{C})$ represents the space of complex lines through the origin in $\mathbb{C}^2$. Analogously $P^1(\mathbb{R})$ represents the space of real lines through the origin in $\mathbb{R}^2$. We can identify $P^1(\mathbb{C})$ with the sphere of Riemann and see $P^1(\mathbb{R})$ as a great circle of S^2. We can also identify $P^1(\mathbb{C})$ with the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by means of the stereographic projection. Then $P^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$.

Taking the complex coordinate $Z = z_2/z_1$ in (3) we obtain the Riccati equations

$$Z' = c(\xi_t) - 2a(\xi_t)Z - b(\xi_t)Z^2 . \tag{3}$$

Let $Z_t(\xi, Z)$ be the solution of the equation (3) with the initial data $Z_0(\xi, Z) = Z$. Then $\Phi_t(\xi, Z) = (\xi_t, Z_t(\xi, Z))$ defines the equation of the flow on $K_{\mathbb{C}}$. Let us denote by r_2 the normalized Lebesgue measure on S^2 and by $m_2 = m_0 \otimes r_2$ the product measure on $K_{\mathbb{C}}$. We shall assume, unless otherwise indicated, all the invariant measures to be positive and normalized. The set $\mathcal{M}_{e,m_0}(K_{\mathbb{C}})$ includes the ergodic measures on $K_{\mathbb{C}}$ which project into m_0.

It is obvious that $K_{\mathbb{R}} = \Omega \times P^1(\mathbb{R})$ is a closed and invariant subset of $K_{\mathbb{C}}$, thus we can consider Φ acting on $K_{\mathbb{R}}$. We can identify $P^1(\mathbb{R})$ with the quotient space $\mathbb{R}/\pi\mathbb{Z}$. Writing the real solutions of (3) in polar symplectic coordinates $\varphi = \cot^{-1}(x_2/x_1)$, $\rho = (x_1^2 + x_2^2)/2$ we obtain

$$\begin{aligned} \varphi' &= f(\xi_t, \varphi) = b(\xi_t)\cos^2\varphi - c(\xi_t)\sin^2\varphi + 2a(\xi_t)\sin\varphi\cos\varphi \\ &= \frac{1}{2}(b(\xi_t) - c(\xi_t)) + a(\xi_t)\sin 2\varphi + \frac{1}{2}(b(\xi_t) + c(\xi_t))\cos 2\varphi, \\ \rho' &= -\frac{\partial f}{\partial \varphi}(\xi_t, \varphi)\rho = (-2a(\xi_t)\cos 2\varphi + (b(\xi_t) + c(\xi_t))\sin 2\varphi)\rho . \end{aligned} \tag{4}$$

Let $\varphi_t(\xi, \varphi)$ be the solution of (4) with initial condition $\varphi_0(\xi, \varphi) = \varphi$. The mapping $\Phi_t(\xi, \varphi) = (\xi_t, \varphi_t(\xi, \varphi))$ is precisely the restriction of the flow to $K_{\mathbb{R}}$. The relation $X = \cot\varphi$ gives us the change between the systems of coordinates that we have introduced. We will denote by r_1 the normalized Lebesgue measure on $P^1(\mathbb{R})$ and by $m_1 = m_0 \otimes r_1$ the product measure on $K_{\mathbb{R}}$. The set $\mathcal{M}_{e,m_0}(K_{\mathbb{R}})$ includes the ergodic measures on $K_{\mathbb{R}}$ which project into m_0.

We say that a real solution $\mathbf{x}(t)$ of the equation (1) has characteristic exponent γ_0 when $t \to \infty$, (*resp.* $t \to -\infty$) if $\overline{\lim}_{t\to\infty}(1/t)\ln|\mathbf{x}(t)| = \gamma_0$, (*resp.* $\overline{\lim}_{t\to-\infty}(1/t)\ln|\mathbf{x}(t)| = \gamma_0$). The maximum characteristic exponent for almost every $\xi \in \Omega$ is called the Lyapunov exponent of (3). In [11] we find the following equivalent definition:

$$\gamma = \sup\left\{-\frac{1}{2}\int_{K_{\mathbb{R}}} \frac{\partial f}{\partial \varphi} d\nu \mid \nu \in \mathcal{M}_{e,m_0}(K_{\mathbb{R}})\right\} .$$

In the present paper we investigate the spectral problems for the self-adjoint operators

$$\mathcal{L}_\xi \mathbf{z} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} [\mathbf{z}' - S(\xi_t)\mathbf{z}] = E\mathbf{z} \tag{5}$$

defined in the domain

$$D = \{\mathbf{z} |\ \mathbf{z} \text{ is absolutely continuous and } \mathbf{z},\ \mathbf{z}' \in L^2(\mathbb{R}, \mathbb{C}^2)\} \subset L^2(\mathbb{R}, \mathbb{C}^2)\ .$$

The spectral classification depends on the behaviour of the solutions for the linear systems

$$\mathbf{z}' = \left[S(\xi_t) + \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \right] \mathbf{z} = S(\xi_t, E)\mathbf{z}\ . \tag{6}$$

There is a closed set σ with spectrum $\sigma(\mathcal{L}_\xi) = \sigma$ for every $\xi \in \Omega$. This set is not necessarely bounded below and may have a quite complicated form.

The basic spectral theory for the self-adjoint operators (5) is described in Johnson [13] and Johnson–Giachetti [14]. The existence of the spectral measure, the relation beetwen resolvent and exponential dichotomy and the behaviour of the Floquet exponent on the complex plane and specially on the spectrum are essential topics treated in these papers. Besides, they are our starting point.

We concentrate our attention on the set $\mathbf{A}$ of those values of the parameter E where the Lyapunov exponent of (6) vanishes and extend the known theory for the usual self-adjoint operators defined by a second order differential equation, which characterizes the absolutely continuous spectrum. Ishii [10] and Pastur [20] proved that the absolutely continuous part of the spectral measure vanishes on the set $\{E |\ \gamma(E) > 0\}$. For our problem this is a consequence of Proposition 3.14 of [14]. Kotani [16] shows that such a measure does not degenerate on $\mathbf{A}$. These results together identify the absolutely continuous spectrum with the essential closure of $\mathbf{A}$. In Section 2 we adapt Kotani's theory to the family of self-adjoint operators (5). We make use of classical properties of Herglotz functions and our strategy of proof is closer to that contained in Deift–Simon [6] or Simon [23]. Even though we follow a known argument line, we detail most of the proofs because the conclusions will be essential for us in the next section.

Section 3 deals with the ergodic structure of the projective flows defined by (6). We show that for almost every E with Lyapunov exponent $\gamma(E) = 0$ the flow $(K_{\mathbb{R}}, \Phi^E)$ admits an invariant measure μ equivalent to m_1 with square integrable density function. Moreover if $d\mu = p dm_1$ and $p = 1/q$ we find that q takes a quadratic expression on the fibers

$$q(\xi, \varphi) = A(\xi) \cos^2 \varphi + B(\xi) \sin^2 \varphi + 2C(\xi) \sin \varphi \cos \varphi$$

with measurable coefficients A, B, C on the base. This is the type of measure that we call linear. The ergodic structure of a projective flow with linear invariant measures is described in Alonso–Obaya [2].

Actually the above theorem was proved in Obaya–Paramio [18] for a second order linear equation. Here we give a different proof based on the techniques of [2] which associate such measures with the complex ergodic sheets of the projective flow. Our argument can be extended to the second order equation and connects ergodic relations of [18] with their equivalent of spectral type given in [6], [15] or [16].

The existence of absolutely continuous invariant measures in the projective bundle has important dynamical consequences but cannot be qualified as a trivial question. Of course, with bounded solutions there exist invariant measures with continuous density function. However a large class of linear systems is known whose projective flow is uniquely ergodic with a unique singular measure. (See Novo–Obaya [17].)

2. Spectral theory

The symbol $\mathbb{C}_+$ will stand for the half plane $\Im z > 0$. By Herglotz functions we mean every holomorphic function mapping $\mathbb{C}_+$ into $\mathbb{C}_+$. We start by recalling some known results concerning Herglotz functions that will be useful for us in what follows. Their proofs can be found in Duren [7].

If h is a Herglotz function then its non-tangential limit from the upper-half plane $h(x) = \lim_{z\to x, n.t.} h(z)$ exists for almost every $x \in \mathbb{R}$. In particular if I is an open interval of $\mathbb{R}$ and $\Re\, h(x) = 0$ for almost every $x \in I$ then h has an analytic continuation through I.

THEOREM 2.1: *The function $h\colon \mathbb{C}_+ \to \mathbb{C}$ is Herglotz if and only if there exist real numbers α, β with $\beta \geq 0$ and a positive measure τ on $\mathbb{R}$ with $1/(1+x^2) \in L^1(\mathbb{R}, \tau)$ such that*

$$h(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left\{ \frac{1}{x-z} - \frac{x}{1+x^2} \right\} d\tau(x).$$

Moreover

(i) *The absolutely continuous part $\tau_{a.c.}$ of τ has density function $\tau_{a.c.}(x) = \lim_{\epsilon\to 0^+} \Im\, h(x+i\epsilon)/\pi$.*

(ii) *The singular part τ_{sing} of τ is concentrated in*

$$D = \{x \mid \limsup_{\epsilon\to 0^+} \Im\, h(x+i\epsilon) = \infty\}.$$

We return to the complex projective bundle. The projective flow contains enough information to understand the behaviour of the solutions of (2).

Definition 2.2: Let M be an invariant subset of $K_{\mathbb{C}}$, $\pi_1: M \to \Omega$ the projection on the base. We say that M is an **ergodic sheet** for the measure m_0, if there is an invariant subset Ω_0 with $m_0(\Omega_0) = 1$ such that

(i) $\operatorname{card}(\pi_1^{-1}(\xi)) = 1$ for every $\xi \in \Omega_0$.

(ii) The map $\Omega_0 \to P^1(\mathbb{R})$, $\xi \to \pi_1^{-1}(\xi)$ is $\mathcal{A}_0$-measurable.

If M is an ergodic sheet of $(K_{\mathbb{C}}, \Phi)$, then $W = \Pi^{-1}(M)$ is a one-dimensional invariant subbundle of $V_{\mathbb{C}}$ which intersects for almost every $\xi \in \Omega$ the fiber $\{\xi\} \times \mathbb{C}^2$ in a complete line through the origin.

Let us assume that $\Omega_0 \subset \Omega$ is an invariant subset with $m_0(\Omega_0) = 1$ and $M = \{(\xi, Z(\xi)) \mid \xi \in \Omega_0\}$ is an ergodic sheet. The flow $(K_{\mathbb{C}}, \Phi)$ possesses an ergodic measure ν defined by

$$\int_{K_{\mathbb{C}}} f\, d\nu = \int_{\Omega} f(\xi, Z(\xi))\, dm_0$$

for every $f \in C(K_{\mathbb{C}})$ which is concentrated in M, i.e. $\nu(M) = 1$, and projects into m_0.

Let $\mathbf{x}(t, \xi, \varphi)$ be the solution of the linear systems (2) along the trajectory that passes through ξ with initial data $x_2(0, \xi, \varphi) + i x_1(0, \xi, \varphi) = \exp(i\varphi)$. Then $U(t, \xi) = (\mathbf{x}(t, \xi, \pi/2), \mathbf{x}(t, \xi, 0))$ defines the fundamental matrix of the above (2) with $U(0, \xi) = \mathrm{Id}$.

Definition 2.3: We say that the equations (2) admit **exponential dichotomy** if there is a splitting $V_{\mathbb{C}} = W_+ \oplus W_-$ where W_+ and W_- are invariant one-dimensional closed subbundles and positive constants C, β such that

(1) $||U(t, \xi)\mathbf{w}|| \le Ce^{-\beta t}$ for $t \in \mathbb{R}$, $(\xi, \mathbf{w}) \in W_+$.

(2) $||U(t, \xi)\mathbf{w}|| \le Ce^{\beta t}$ for $t \in \mathbb{R}$, $(\xi, \mathbf{w}) \in W_-$.

Let us consider the family of linear systems (6). Of course the existence or nonexistence of exponential dichotomy depends on E and is related to the spectral problem for the self-adjoint operators (5).

Since (Ω, Ξ) is a minimal flow then the spectrum of $\mathcal{L}_\xi$ is independent of $\xi \in \Omega$. Besides, the resolvent can be characterized via exponential dichotomy. (See [14].)

THEOREM 2.4: *The spectrum of the full-line operator $\mathcal{L}_\xi$ is a closed subset of $\mathbb{R}$. Moreover the following facts are equivalent:*

(i) *$E \in \mathbb{C}$ is in the resolvent of $\mathcal{L}_\xi$.*

(ii) *The linear systems $\mathbf{z}' = S(\xi_t, E)\mathbf{z}$ admit exponential dichotomy.*

Now, if E belongs to the resolvent of $\mathcal{L}_\xi$ and we have the splitting $V_{\mathbb{C}} = W_+^E \oplus W_-^E$, then $M_\pm^E = \Pi(W_\pm^E) = \{(\xi, m_\pm(\xi, E)) | \ \xi \in \Omega\}$ define closed ergodic sheets. In this context we can say that the Weyl–Titchmarsh functions are a consequence of the exponential dichotomy. The functions $m_\pm(\xi, E)$ are jointly continuous in both variables. (See Sacker–Sell [21].) For every $\xi \in \Omega$ fixed, the functions $E \to m_\pm(\xi, E)$ are analytic in the domain $\Im E \neq 0$. (See Johnson [12].) Moreover one has that $\pm\Im m_\pm(\xi, E)\Im E > 0$ and $m_\pm(\xi, \overline{E}) = \overline{m_\pm(\xi, E)}$. Thus, for each $\xi \in \Omega$ we see that $\pm m_\pm(\xi, E)$ are Herglotz functions on $\Im E > 0$ and their non-tangential limits from the upper half-plane exist for almost every $E_0 \in \mathbb{R}$. Then we denote by $m_\pm(\xi, E_0) = \lim_{E \to E_0, n.t.} m_\pm(\xi, E)$ the value of this limit.

The spectral theory for the self-adjoint operators (5), described in [13] and [14], follows from the same arguments used for the full-line operators defined by a second order linear equation in the limit-point case at $\pm\infty$. (See Coddington–Levinson [5].)

THEOREM 2.5: *Let $\xi \in \Omega$. There exists a spectral matrix $Q_\xi(t)$ and an unitary isomorphism U_ξ: $L^2(\mathbb{R}, \mathbb{C}^2, r) \to L^2(\mathbb{R}, \mathbb{C}^2, dQ_\xi)$ which transforms $\mathcal{L}_\xi$ into M, being*

$$\begin{array}{rcl} M\colon D_\xi \subset L^2(\mathbb{R}, \mathbb{C}^2, dQ_\xi) & \longrightarrow & L^2(\mathbb{R}, \mathbb{C}^2, dQ_\xi) \\ g & \longrightarrow & M(g)(E) = Eg(E) \end{array}$$

the operator of multiplication by E.

For such $\xi \in \Omega$ fixed it is known that $Q_\xi = (Q_\xi^{i,j})_{i,j=1,2}$ is a non-decreasing Hermitian matrix of bounded variation on every finite interval. Moreover for every Borel subset $A \subset \mathbb{R}$ we have

$$|dQ_\xi^{1,2}(A)|^2 = |dQ_\xi^{2,1}(A)|^2 \leq |dQ_\xi^{1,1}(A)||dQ_\xi^{2,2}(A)|$$

and thus we can understand the trace of dQ_ξ as the spectral measure. In particular $\tau = \operatorname{tr} dQ_\xi$ increases exactly on the spectrum σ.

The Floquet exponent $\omega(E) = -\gamma(E) + i\alpha(E)$ is defined as a complex function whose real and imaginary parts are the negative Lyapunov exponent and the rotation number respectively. It is an analytic function on $\Im E \neq 0$ where it admits an ergodic representation

$$\omega(E) = \pm \int_\Omega [a(\xi) + (E + b(\xi))m_\pm(\xi, E)]\, dm_0 \tag{7}$$

which comes from an extension of $\pm\rho'/2\rho + i\varphi'$ to the entire complex plane. Moreover one has

$$\omega'(E) = \int_\Omega \frac{1 + m_-(\xi, E)m_+(\xi, E)}{m_-(\xi, E) - m_+(\xi, E)}\, dm_0\,. \tag{8}$$

Relations (7) and (8) show that $\omega(E)$ and $\omega'(E)$ are Herglotz functions on $\Im E > 0$ and assure the extension of $\omega(E)$ to $\mathbb{R}$. On the real axis $\alpha(E)$ is a continuous function which increases on the spectrum and is constant on the resolvent. It is known that

$$\Im\omega'(E) = \Im E \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\alpha(x)}{|x - E|^2}. \tag{9}$$

We now return to the Riccati equations (3) and analyze the convergence of $m_\pm(\xi, E_0) = \lim_{E\to E_0} m_\pm(\xi, E)$. As a consequence of Fubini's theorem we find that for almost every $E_0 \in \mathbb{R}$ the above limits exist almost everywhere in Ω. Given a point of convergence $E_0 \in \mathbb{R}$ the set of $\xi's$ with real limit and the set of $\xi's$ with complex (but not real) limit are invariant and disjoint and one of them defines an invariant subset Ω_0 with complete measure. Moreover $M_\pm = \{(\xi, m_\pm(\xi, E_0)) \mid \xi \in \Omega_0\}$ are measurable ergodic sheets.

Pastur [20] and Ishii [10] showed, for the usual spectral problem defined by a second order linear equation, that the absolutely continuous part of the spectral measure vanishes on the set $\{E \mid \gamma(E) > 0\}$. For our problem, this derives from the following result, stated in [14] for the linear systems (6). (See also [6].)

PROPOSITION 2.6: *Let $E_0 \in \mathbb{R}$ with $\gamma(E_0) > 0$. Then there exist $m_\pm(\xi, E_0) = \lim_{E\to E_0, n.t.} m_\pm(\xi, E)$ and they belong to $\mathbb{R} \cup \{\infty\}$ for almost every $\xi \in \Omega$.*

We mention that adapting Kotani's arguments from [16] we can refine the previous information when the Lyapunov exponent vanishes. From now on we denote $\mathbf{A} = \{E_0 \in \mathbb{R} \mid \gamma(E_0) = 0\}$. Let $\mathbf{A}_+$ be the set of these E_0 where there exist $m_+(\xi, E_0) = \lim_{E\to E_0, n.t.} m_+(\xi, E)$ with $\Im m_+(\xi, E_0) \neq 0$ for almost every $\xi \in \Omega$. We define $\mathbf{A}_-$ in a similar way and take $\mathbf{A}_1 = \mathbf{A}_+ \cup \mathbf{A}_-$. It follows from Proposition 2.6 that $\mathbf{A}_1 \subset \mathbf{A}$.

PROPOSITION 2.7: *There is a subset $\mathbf{A}_2 \subset \mathbf{A}_1$ with $r(\mathbf{A} - \mathbf{A}_2) = 0$ such that if $E_0 \in \mathbf{A}_2$ then the maps $\dfrac{1}{\Im m_\pm(\xi, E_0)}, \Im m_\pm(\xi, E_0), \dfrac{\Re^2 m_\pm(\xi, E_0)}{\Im m_\pm(\xi, E_0)}$ belong to $L^1(\Omega, m_0)$.*

Proof: We remind the reader that for almost every $E_0 \in \mathbb{R}$ the non-tangential limits of $\pm m_\pm(\xi, E)$ exist for almost every $\xi \in \Omega$.

On the other hand, if $E_0 \in \mathbb{R}$ and $\epsilon > 0$ then

$$\omega'(E_0 + i\epsilon) = \frac{\partial \gamma}{\partial E_0}(E_0 + i\epsilon) - i\frac{\partial \gamma}{\partial \epsilon}(E_0 + i\epsilon) .$$

If $\gamma(E_0) = 0$ then

$$\frac{-\Re\omega(E_0 + i\epsilon)}{\epsilon} = \frac{\gamma(E_0 + i\epsilon) - \gamma(E_0)}{\epsilon} = \frac{\partial \gamma}{\partial \epsilon}(E_0 + i\epsilon')$$

with $0 < \epsilon' < \epsilon$. We take $E_0 \in \mathbf{A}$ a point of differentiability of the rotation number. We recall the expression (9), now a standard argument of measure theory allows to derive

$$\lim_{\epsilon \to 0^+} \frac{-\Re\omega(E_0 + i\epsilon)}{\epsilon} = \lim_{\epsilon \to 0^+} \frac{\partial \gamma}{\partial \epsilon}(E_0 + i\epsilon) = \alpha'(E_0) < \infty. \tag{10}$$

We represent by $\mathbf{A}_2$ the subset of $\mathbf{A}$ where all the above limits exist. According to the previous remarks it is obvious that $r(\mathbf{A} - \mathbf{A}_2) = 0$.

When $\Im E > 0$, the functions $m_\pm(\xi_t, E)$ satisfy the differential equation

$$Z' = (-E + c(\xi_t)) - 2a(\xi_t)Z - (E + b(\xi_t))Z^2$$

and taking the imaginary part we obtain

$$\begin{aligned}\Im m'_\pm(\xi_t, E) = & -\Im E - 2a(\xi_t)\Im m_\pm(\xi_t, E) \\ & -\Im E(\Re^2 m_\pm(\xi_t, E) - \Im^2 m_\pm(\xi_t, E)) \\ & -2(\Re E + b(\xi_t))\Im m_\pm(\xi_t, E)\Re m_\pm(\xi_t, E);\end{aligned}$$

then

$$\begin{aligned}&\frac{\Im m'_\pm(\xi_t, E)}{\Im m_\pm(\xi_t, E)} + \Im E\left(\frac{1 + |m_\pm(\xi_t, E)|^2}{\Im m_\pm(\xi_t, E)}\right) = \\ &\quad -2(a(\xi_t) + (\Re E + b(\xi_t))\Re m_\pm(\xi_t, E) - \Im E \Im m_\pm(\xi_t, E)) ,\end{aligned}$$

and in consequence

$$\begin{aligned}&\frac{1}{2T}\int_{-T}^{T} \frac{\Im m'_\pm(\xi_t, E)}{\Im m_\pm(\xi_t, E)}\, dt + \frac{\Im E}{2T}\int_{-T}^{T} \frac{1 + |m_\pm(\xi_t, E)|^2}{\Im m_\pm(\xi_t, E)}\, dt = \\ &\quad -\frac{1}{T}\int_{-T}^{T} (a(\xi_t) + (\Re E + b(\xi_t))\Re m_\pm(\xi_t, E) - \Im E \Im m_\pm(\xi_t, E))\, dt .\end{aligned}$$

We take limits as $T \to \infty$ and use the equality (7). It follows from Birkhoff's ergodic theorem that

$$\Im E \int_\Omega \frac{1+|m_\pm(\xi,E)|^2}{|\Im m_\pm(\xi,E)|}\, dm_0 = -2\Re\omega(E). \tag{11}$$

This is the extension of a known Kotani's equality for the second order linear equation.

Now we choose $E_0 \in \mathbf{A}_2$ and set $E = E_0 + i\epsilon$. We take limits as ϵ tends to 0 and use the equality (10). It follows Fatou's lemma that

$$\int_\Omega \frac{1+|m_\pm(\xi,E_0)|^2}{|\Im m_\pm(\xi,E_0)|}\, dm_0 \le \lim_{\epsilon\to 0^+} \frac{-2\Re\omega(E_0+i\epsilon)}{\epsilon} = 2\alpha'(E_0)$$

which shows that

$$\frac{1+|m_\pm(\xi,E_0)|^2}{|\Im m_\pm(\xi,E_0)|} \in L^1(\Omega, m_0)\,.$$

In particular this means that $\Im m_\pm(\xi,E_0) \neq 0$ for almost every $\xi \in \Omega$. ∎

For $\Im E \neq 0$ we introduce the matrix

$$G(\xi,E) = \begin{pmatrix} \dfrac{1}{m_-(\xi,E)-m_+(\xi,E)} & \dfrac{1}{2}\dfrac{m_-(\xi,E)+m_+(\xi,E)}{m_+(\xi,E)-m_-(\xi,E)} \\ \dfrac{1}{2}\dfrac{m_-(\xi,E)+m_+(\xi,E)}{m_+(\xi,E)-m_-(\xi,E)} & \dfrac{m_-(\xi,E)m_+(\xi,E)}{m_-(\xi,E)-m_+(\xi,E)} \end{pmatrix} \tag{12}$$

which is related to the Green kernel for the self-adjoint operator (5) at ξ. Moreover, if $G = (G^{i,j})_{i,j=1,2}$ then

$$\Im G^{i,j}(\xi,E) = \Im E \int_{-\infty}^{\infty} \frac{(-1)^{i+j}\, dQ_\xi^{i,j}(x)}{|x-E|^2}\,. \tag{13}$$

For $E_0 \in (\mathbb{R} - \mathbf{A}) \cup \mathbf{A}_2$ we understand $G(\xi,E_0) = \lim_{E\to E_0, n.t.} G(\xi,E)$.

PROPOSITION 2.8: *If $E_0 \in \mathbf{A}_2$, then*

(i) $\overline{m_+(\xi,E_0)} = m_-(\xi,E_0)$,

(ii) $\Re G(\xi,E_0) = 0$,

for almost every $\xi \in \Omega$.

Proof: For $\Im E > 0$ we consider the functions

$$\begin{aligned} I_1(\xi,E) &= \frac{1}{4}\left(\frac{1+|m_+(\xi,E)|^2}{\Im m_+(\xi,E)} - \frac{1+|m_-(\xi,E)|^2}{\Im m_-(\xi,E)}\right) \\ &= \frac{\Im m_-(\xi,E)(1+|m_+(\xi,E)|^2) - \Im m_+(\xi,E)(1+|m_-(\xi,E)|^2)}{4\Im m_+(\xi,E)\Im m_-(\xi,E)} \end{aligned}$$

and

$$I_2(\xi,E) = \Im\left(\frac{1+m_-(\xi,E)m_+(\xi,E)}{m_-(\xi,E)-m_+(\xi,E)}\right)$$
$$= \frac{\Im m_+(\xi,E)(1+|m_-(\xi,E)|^2) - \Im m_-(\xi,E)(1+|m_+(\xi,E)|^2)}{|m_-(\xi,E)-m_+(\xi,E)|^2}.$$

Notice that $0 < I_2(\xi,E) \le I_1(\xi,E)$ for every $\xi \in \Omega$, $E \in \mathbb{C}_+$. First from (11) and (8) we have

$$\frac{-\Re\omega(E_0+i\epsilon)}{\epsilon} - \Im\omega'(E_0+i\epsilon) = \int_\Omega [I_1(\xi,E_0+i\epsilon) - I_2(\xi,E_0+i\epsilon)]\,dm_0\,.$$

Moreover

$$\frac{-\Re\omega(E_0+i\epsilon)}{\epsilon} - \Im\omega'(E_0+i\epsilon) = \frac{\partial\gamma}{\partial\epsilon}(E_0+i\epsilon') - \frac{\partial\gamma}{\partial\epsilon}(E_0+i\epsilon)$$

with $0 < \epsilon' < \epsilon$. Taking limits when ϵ tends to 0 and using the Fatou lemma we get

$$I_1(\xi,E_0) - I_2(\xi,E_0) \ge 0$$

and

$$\int_\Omega [I_1(\xi,E_0) - I_2(\xi,E_0)]\,dm_0 = 0\,.$$

We derive that $I_1(\xi,E_0) = I_2(\xi,E_0)$ almost everywhere which allows to conclude that $\Re m_+(\xi,E_0) = \Re m_-(\xi,E_0)$ and $\Im m_+(\xi,E_0) = -\Im m_-(\xi,E_0)$ for almost every $\xi \in \Omega$. This proves the statement (i). By evaluating (12) we immediatly obtain (ii). ∎

Now, applying Theorem 2.1 in the relation (13) we see that the absolutely continuous part $\tau_{a.c.}$ of the spectral measure $\tau = \operatorname{tr} dQ_\xi$ has density function $(1+|m_+(\xi,E)|^2)/2\pi\Im m_+(\xi,E)$. Thus from Proposition 2.7 we conclude

THEOREM 2.9: *The absolutely continuous spectrum of $\mathcal{L}_\xi$ agrees with the essential closure of* $\mathbf{A}$ *for almost every* $\xi \in \Omega$.

Thus $\sigma_{a.c.} = \overline{\mathbf{A}}^r = \{E \in \mathbb{R} | \ (\forall\epsilon > 0)\ r[(E_0-\epsilon, E_0+\epsilon) \cap \mathbf{A}] > 0\}$. There exists a subset $\mathbf{B} \subset \mathbb{R}$ with $r(\mathbf{B}) = 0$ such that $\sigma_{a.c.} \subset \mathbf{A} \cup \mathbf{B}$. Moreover if $r(\sigma - \mathbf{A}) = 0$ one has the important reflectionless condition $\Re G(\xi,E_0) = 0$ for almost every $(\xi,E_0) \in \Omega \times \sigma$.

A careful analysis of the non-tangential limits of the Weyl–Titschmarch functions on $\mathbf{A}$ leads us to state

THEOREM 2.10: *If $\gamma(E) = 0$ on an open interval I of $\mathbb{R}$ then the spectral measure of $\mathcal{L}_\xi$ is purely absolutely continuous on I for almost every $\xi \in \Omega$.*

Proof: We see that $\operatorname{tr} G(\xi, E)$ is a Herglotz function whose real part vanishes on I. It has an analytic continuation through I with positive imaginary part. Let $D = \{E_0 \in \mathbb{R} |\ \limsup_{\epsilon \to 0^+} \Im \operatorname{tr} G(\xi, E_0 + i\epsilon) = \infty\}$; then $D \cap I = \emptyset$ and hence $\sigma_{sing}(I) = 0$. ∎

Notice that the above argument also applies to $1/(m_-(\xi, E) - m_+(\xi, E))$. Combining both facts we guarantee the limits $m_\pm(\xi, E_0) = \lim_{\epsilon \to 0^+, n.t.} m_\pm(\xi, E_0 + i\epsilon)$ for every $E_0 \in I$. This immediately gives the sequence of inclusions $\operatorname{int}(\mathbf{A}) \subset \mathbf{A}_2 \subset \mathbf{A}_1 \subset \mathbf{A}$.

3. Linear invariant measures

Let m_0 be a fixed ergodic measure on Ω, $\mathcal{A}_0$ the completion of the σ-algebra of the Borel sets with respect to m_0. The symbol $m_1 = m_0 \otimes r_1$ stands for the complete product measure on the corresponding σ-algebra $\mathcal{A}_1$ of $\Omega \times P^1(\mathbb{R})$. Similarly, the symbol $m_2 = m_0 \otimes r_2$ will denote the complete product measure on the corresponding σ-algebra $\mathcal{A}_2$ of $\Omega \times P^1(\mathbb{C})$.

Let us consider the linear systems (2) whose real and complex projective flows are respectively defined by the equations (4) and (3). It is known that if the flow $(K_\mathbb{R}, \Phi)$ admits an invariant measure which is absolutely continuous with respect to m_1, then $(K_\mathbb{C}, \Phi)$ admits an invariant measure which is absolutely continuous with respect m_2. This situation only occurs when the Lyapunov exponent vanishes.

In this section we concentrate our attention on the following remarkable class of absolutely continuous invariant measures.

Definition 3.1: Let μ be an invariant measure on $K_\mathbb{R}$ with $d\mu = p dm_1$. We say that μ is a **linear invariant measure** if there are measurable functions $A, B, C \colon \Omega \to \mathbb{R}$ and an invariant subset $\Omega_0 \subset \Omega$ with $m_0(\Omega_0) = 1$ such that $p(\xi, \varphi) = 1/q(\xi, \varphi)$ and

$$q(\xi, \varphi) = A(\xi) \cos^2 \varphi + B(\xi) \sin^2 \varphi + 2C(\xi) \sin \varphi \cos \varphi \tag{14}$$

for every $(\xi, \varphi) \in \Omega_0 \times P^1(\mathbb{R})$.

From these coefficients A, B, C we introduce the map

$$\mathbf{X} = (A, B, C)^t \colon \Omega \to \mathbb{R}^3.$$

It is known that there exists an invariant subset $\Omega_0 \subset \Omega$ with $m_0(\Omega_0) = 1$ such that the map $\mathbf{X}_\xi: \mathbb{R} \to \mathbb{R}^3$, $t \to \mathbf{X}(\xi_t)$ is a C^1 map with $A(\xi)B(\xi) - C^2(\xi) = 1$, for every $\xi \in \Omega_0$. (See [2].)

Let $\tilde{S} \in C(\Omega, L(\mathbb{R}^3))$. We say that $\mathbf{X}$ is a solution along the flow for m_0 of the linear systems $\mathbf{X}' = \tilde{S}(\xi_t)\mathbf{X}$ if there is an invariant subset $\Omega_0' \subset \Omega$ with $m_0(\Omega_0') = 1$ such that $\mathbf{X}'(\xi_t) = \tilde{S}(\xi_t)\mathbf{X}(\xi_t)$ for every $(\xi, t) \in \Omega_0' \times \mathbb{R}$.

The above function q is a positive solution of the functional equation

$$q(\Phi_t(\xi,\varphi)) = q(\xi,\varphi)\exp\left\{\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))\,ds\right\}. \tag{15}$$

Every function on $K_{\mathbb{R}}$ which takes the form (14) is said quadratic on the fibers. The quadratic on the fibers solutions of (15) are directly related to the ergodic sheets of the projective flows. The following facts are proved in [2].

Proposition 3.2: *Let us consider the measurable map $\mathbf{X} = (A, B, C)^t: \Omega \to \mathbb{R}^3$. The following statements are equivalent:*

(i) *The function*

$$q(\xi,\varphi) = A(\xi)\cos^2\varphi + B(\xi)\sin^2\varphi + 2C(\xi)\sin\varphi\cos\varphi$$

is a solution of the functional equation (15).

(ii) $\mathbf{X}$ *is a solution along the flow of the systems*

$$\mathbf{X}' = \begin{pmatrix} 2a(\xi_t) & 0 & -2b(\xi_t) \\ 0 & -2a(\xi_t) & -2c(\xi_t) \\ -c(\xi_t) & -b(\xi_t) & 0 \end{pmatrix} \mathbf{X}. \tag{16}$$

The concept of solution along the flow of linear systems associates the coefficients A, B, C of q with invariant subsets of the projective flow.

Theorem 3.3: *Let $\Omega_0 \subset \Omega$ be an invariant subset with $m_0(\Omega_0) = 1$.*

(i) *If $M = \{(\xi, X(\xi) + iY(\xi)) \mid \xi \in \Omega_0\}$ is a complex ergodic sheet of $(K_{\mathbb{C}}, \Phi)$ with $Y(\xi) > 0$ for every $\xi \in \Omega_0$ and we define*

$$p(\xi,\varphi) = \left(\frac{1}{Y(\xi)}\cos^2\varphi + \frac{X^2(\xi)+Y^2(\xi)}{Y(\xi)}\sin^2\varphi - \frac{2X(\xi)}{Y(\xi)}\sin\varphi\cos\varphi\right)^{-1}$$

for every $(\xi,\varphi) \in \Omega_0 \times P^1(\mathbb{R})$, then $d\mu = p\,dm_1$ is a linear invariant measure.

(ii) *If* $M_i = \{(\xi, X_i(\xi)) \mid \xi \in \Omega_0\}$ $i = 1, 2$ *are two real ergodic sheets of* $(K_{\mathbb{C}}, \Phi)$ *and we define*

$$A(\xi) = \frac{2}{X_1(\xi) - X_2(\xi)},\ B(\xi) = \frac{2X_1(\xi)X_2(\xi)}{X_1(\xi) - X_2(\xi)},\ C(\xi) = \frac{X_2(\xi) + X_1(\xi)}{X_2(\xi) - X_1(\xi)},$$

then

$$q(\xi, \varphi) = A(\xi)\cos^2\varphi + B(\xi)\sin^2\varphi + 2C(\xi)\sin\varphi\cos\varphi$$

is a measurable solution of (15). *Moreover,* $A(\xi)B(\xi) - C^2(\xi) = -1$ *for every* $\xi \in \Omega_0$.

Conversely, complex and real ergodic sheets can be recovered from the measurable solutions of the functional equation (15). These solutions contain essential information for understanding the topological and measurable structures of the complex projective flow.

For each $T > 0$ we introduce the function

$$p_T(\xi, \varphi) = \frac{1}{2T}\int_{-T}^{T} \exp\left\{\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi, \varphi)\, ds\right\} dt.$$

Propositions 2.3 and 2.4 of [18] assert that $(K_{\mathbb{R}}, \Phi)$ admits an invariant measure equivalent to m_1 if and only if the limit $p(\xi, \varphi) = \lim_{T\to\infty} p_T(\xi, \varphi)$ exists and is a positive real number for almost every $(\xi, \varphi) \in K_{\mathbb{R}}$. In this case $d\mu = p\, dm_1$ is an invariant measure equivalent to m_1.

The following maps

$$\begin{array}{rlcl} d_T\colon & \Omega \times P^1(\mathbb{R}) \times P^1(\mathbb{R}) & \longrightarrow & P^1(\mathbb{R}) \\ & (\xi, \varphi_1, \varphi_2) & \longrightarrow & \frac{1}{2T}\int_{-T}^{T}\{\varphi(t, \xi, \varphi_2) - \varphi(t, \xi, \varphi_1)\}\, dt \end{array}$$

evaluate the L^1-oscillations of the projective flow on $K_{\mathbb{R}}$. The existence of absolutely continuous invariant measures has important dynamical consequences. We analyze the evolution of the family $\{d_T\}_{T>0}$ along the time.

PROPOSITION 3.4: *Let assume that* $(K_{\mathbb{R}}, \Phi)$ *admits an invariant measure equivalent to* m_1. *We take* $p(\xi, \varphi) = \lim_{T\to\infty} p_T(\xi, \varphi)$ *for almost every* $(\xi, \varphi) \in K_{\mathbb{R}}$. *There is an invariant subset* $\Omega_0 \subset \Omega$ *with* $m_0(\Omega_0) = 1$ *such that if* $\xi \in \Omega_0$ *then* $\{d_T(\xi, \varphi_1, \varphi_2)\}_{T>0}$ *uniformly converges on* φ_1, $\varphi_2 \in P^1(\mathbb{R})$ *when* $T \to \infty$. *Moreover*

$$d(\xi, \varphi_1, \varphi_2) = \lim_{T\to\infty} d_T(\xi, \varphi_1, \varphi_2) = \int_{\varphi_1}^{\varphi_2} p(\xi, \varphi)\, d\varphi\,.$$

Proof: Let $T > 0$, it is obvious that $p_T \in C(K_{\mathbb{R}})$ and $\int_{P^1(\mathbb{R})} p_T(\xi,\varphi)\,d\varphi = 1$ for every $\xi \in \Omega$. Notice that

$$\begin{aligned} d_T(\xi,\varphi_1,\varphi_2) &= \frac{1}{2T}\int_{-T}^{T}\{\varphi(t,\xi,\varphi_2) - \varphi(t,\xi,\varphi_1)\}\,dt \\ &= \frac{1}{2T}\int_{-T}^{T}\int_{\varphi_1}^{\varphi_2} \exp\left\{\int_0^t \frac{\partial f}{\partial\varphi}(\Phi_s(\xi,\varphi))\,ds\right\} d\varphi\,dt \\ &= \int_{\varphi_1}^{\varphi_2} p_T(\xi,\varphi)\,d\varphi. \end{aligned}$$

By ergodicity on the base it is known that there is an invariant subset $\Omega_0 \subset \Omega$ with $m_0(\Omega_0) = 1$ such that if $\xi \in \Omega_0$ then $\int_{P^1(\mathbb{R})} p(\xi,\varphi)\,d\varphi = 1$ and $p(\xi,\varphi) = \lim_{T\to\infty} p_T(\xi,\varphi)$ for almost every $\varphi \in P^1(\mathbb{R})$.

Let us fix $\xi \in \Omega_0$. We take a measurable subset $I \subset P^1(\mathbb{R})$ and $J = P^1(\mathbb{R}) - I$. It follows from Fatou's lemma that

$$\begin{aligned} \int_I p(\xi,\varphi)\,d\varphi &\le \liminf_{T\to\infty}\int_I p_T(\xi,\varphi)\,d\varphi \le \limsup_{T\to\infty}\int_I p_T(\xi,\varphi)\,d\varphi \\ &\le 1 - \liminf_{T\to\infty}\int_J p_T(\xi,\varphi)\,d\varphi \le 1 - \int_J p(\xi,\varphi)\,d\varphi \\ &= \int_I p(\xi,\varphi)\,d\varphi\,. \end{aligned}$$

In particular, when I stands for the closed arc obtained by moving clockwise from φ_1 to φ_2 on $P^1(\mathbb{R})$ we have

$$d(\xi,\varphi_1,\varphi_2) = \lim_{T\to\infty} d_T(\xi,\varphi_1,\varphi_2) = \int_{\varphi_1}^{\varphi_2} p(\xi,\varphi)\,d\varphi\,.$$

We also deduce as a simple consequence of Egorov's theorem the convergence of $\{p_T(\xi,\varphi)\}$ to $p(\xi,\varphi)$ when $T \to \infty$ in $L^1(P^1(\mathbb{R}), r_1)$-topology. Even this same argument shows the convergence of $\{p_T\}$ to p in the $L^1(K_{\mathbb{R}}, m_1)$-topology.

For $\xi \in \Omega_0$, if $\varphi_1 = 0$, $\varphi_2 = \varphi$ then $d_T(\xi,0,\varphi)$, $d(\xi,0,\varphi)$ are homeomorphisms of $P^1(\mathbb{R})$ preserving the orientation. Arzela–Ascoli's theorem assures the uniform convergence of $\{d_T(\xi,0,\varphi)\}_{T>0}$ to $d(\xi,0,\varphi)$ on $P^1(\mathbb{R})$. Since $d_T(\xi,\varphi_1,\varphi_2) = d_T(\xi,0,\varphi_2) - d_T(\xi,0,\varphi_1)$ then $\{d_T(\xi,\varphi_1,\varphi_2)\}_{T>0}$ also converges uniformly on φ_1, φ_2 when $T \to \infty$. ∎

Let us consider the family of functions

$$q_T(\xi,\varphi) = \frac{1}{2T}\int_{-T}^{T} \exp\left\{-\int_0^t \frac{\partial f}{\partial\varphi}(\Phi_s(\xi,\varphi))\,ds\right\} dt.$$

Propositions 2.3 and 2.4 of [18] assert that $(K_{\mathbb{R}}, \Phi)$ admits an absolutely continuous invariant measure with square integrable density function if and only if the limit $q(\xi,\varphi) = \lim_{T\to\infty} q_T(\xi,\varphi)$ exists and is a positive real number for almost every $(\xi,\varphi) \in K_{\mathbb{R}}$.

In this case, if $\lambda = \int_{K_{\mathbb{R}}} (1/q)\, dm_1$ and $p = 1/(\lambda q)$ then $p \in L^2(K_{\mathbb{R}}, m_1)$ and $d\mu = p dm_1$ is a linear invariant measure. Moreover it follows from Birkhoff's ergodic theorem that

$$\lim_{t\to\infty} \frac{p(\Phi_t(\xi,\varphi))}{t} = \int_{K_{\mathbb{R}}} -\frac{\partial f}{\partial \varphi}(\xi,\varphi) p(\xi,\varphi)\, dm_1 = 0$$

for almost every $(\xi,\varphi) \in K_{\mathbb{R}}$. By other hand, the functions $x_1^2(t,\xi,\varphi)+x_2^2(t,\xi,\varphi)$ and $p(\Phi_t(\xi,\varphi))$ satisfy the same differential equation, which allows us to write $p(\Phi_t(\xi,\varphi)) = p(\xi,\varphi)[x_1^2(t,\xi,\varphi) + x_2^2(t,\xi,\varphi)]$. This provides an invariant subset $\Omega_0 \subset \Omega$ with $m_0(\Omega_0) = 1$ with the following size of the solutions

$$\lim_{t\to\infty} \frac{x_1^2(t,\xi,\varphi) + x_2^2(t,\xi,\varphi)}{t} = 0 \tag{17}$$

for every $\xi \in \Omega_0$.

Let us consider the one-parameter family of linear systems (6). The corresponding equation (15) is different for each E, however we take for granted this dependence and avoid the parameter on the notation. The following statement provides a linear invariant measure for almost every E with null Lyapunov exponent.

THEOREM 3.5: *Let $\mathbf{A}_1$, $\mathbf{A}_2$ be the sets obtained in Section 2. Then*

(i) *If $E_0 \in \mathbf{A}_1$ then $(K_{\mathbb{R}}, \Phi^{E_0})$ admits a linear invariant measure.*

(ii) *If $E_0 \in \mathbf{A}_2$ and q is a quadratic on the fibers solution of (15) then $q \in L^1(K_{\mathbb{R}}, m_1)$.*

(iii) *If $E_0 \in \mathbf{A}_2$ and μ is a linear invariant measure where $d\mu = p dm_1$ then $p \in L^2(K_{\mathbb{R}}, m_1)$.*

Proof: Let $E_0 \in \mathbf{A}_1$. We can suppose that $m_+(\xi, E_0) = \lim_{E\to E_0, n.t.} m_+(\xi, E)$ exists for almost every $\xi \in \Omega$, otherwise we would consider m_-, and define $M_+ = \{(\xi, m_+(\xi, E_0)) |\ \xi \in \Omega\}$. The sets M_+, $\overline{M_+}$ are complex ergodic sheets which provide, according to Theorem 3.3(i), a linear invariant measure μ_0, where

$d\mu_0 = p_0 dm_1$. We introduce

$$G_+(\xi, E_0) = \begin{pmatrix} \dfrac{i}{2\Im m_+(\xi, E_0)} & \dfrac{-\Re m_+(\xi, E_0)i}{2\Im m_+(\xi, E_0)} \\ \dfrac{-\Re m_+(\xi, E_0)i}{2\Im m_+(\xi, E_0)} & \dfrac{|m_+(\xi, E_0)|^2 i}{2\Im m_+(\xi, E_0)} \end{pmatrix}.$$

If $p_0 = 1/q_0$ then q_0 is a quadratic on the fibers function associated to the matrix $2\Im G_+(\xi, E_0)$. This means that if $\mathbf{v}^t(\varphi) = (\cos\varphi, \sin\varphi)$ then

(18) $$q_0(\xi, \varphi) = 2\mathbf{v}^t(\varphi)\Im G_+(\xi, E_0)\mathbf{v}(\varphi).$$

If $E_0 \in \mathbf{A}_2$ then the sets $M_\pm = \{(\xi, m_\pm(\xi, E_0)) \mid \xi \in \Omega\}$ define two complex ergodic sheets with $M_\pm = \overline{M_\mp}$. From Proposition 2.8 we deduce that $G(\xi, E_0) = \lim_{E \to E_0, n.t.} G(\xi, E)$ and both previous definitions agree on $\mathbf{A}_2$. It follows from Proposition 2.7 that

$$\int_\Omega \frac{1 + |m_\pm(\xi, E_0)|^2}{|\Im m_\pm(\xi, E_0)|}\, dm_0 < \infty$$

Since

$$q_0(\xi, \varphi) + q_0\left(\xi, \varphi + \frac{\pi}{2}\right) = \frac{1 + |m_\pm(\xi, E_0)|^2}{|\Im m_\pm(\xi, E_0)|}$$

for every $(\xi, \varphi) \in K_\mathbb{R}$ we obtain that $q_0 \in L^1(K_\mathbb{R}, m_1)$. Moreover

(19) $$||q_0||_1 = \int_\Omega \frac{1 + |m_\pm(\xi, E_0)|^2}{2|\Im m_\pm(\xi, E_0)|}\, dm_0.$$

The ergodic components for m_0 of the flow $(K_\mathbb{C}, \Phi^{E_0})$ are described in [2]. We know that μ_0 need not be the unique linear invariant measure. If $q(\xi, \varphi) = A(\xi)\cos^2\varphi + B(\xi)\sin^2\varphi + 2C(\xi)\sin\varphi\cos\varphi$ is a measurable solution of the equation (15) then the function q/q_0 is invariant under the flow. Besides it is continuous on φ for almost every $\xi \in \Omega$. Let us introduce the functions $k_1(\xi) = \min_{\varphi \in P^1(\mathbb{R})}(q(\xi, \varphi)/q_0(\xi, \varphi))$, $k_2(\xi) = \max_{\varphi \in P^1(\mathbb{R})}(q(\xi, \varphi)/q_0(\xi, \varphi))$. Then $k_1(\xi)$, $k_2(\xi)$ are invariant under the flow and hence by ergodicity on the base there exists an invariant subset $\Omega_0 \subset \Omega$ with $m_0(\Omega_0) = 1$ and real constants k_1, k_2 such that $k_1(\xi) = k_1$, $k_2(\xi) = k_2$ for every $\xi \in \Omega_0$. Thus we obtain that $|q(\xi, \varphi)| \le (|k_2| + |k_1|)q_0(\xi, \varphi)$ and $q \in L^1(K_\mathbb{R}, m_1)$. This completes (ii) .

We next deal with the statement (iii). Let $d\mu = p\, dm_1$ be a linear invariant measure where $p = 1/q$ and $q(\xi, \varphi) = A(\xi)\cos^2\varphi + B(\xi)\sin^2\varphi + 2C(\xi)\sin\varphi\cos\varphi$. Since

$$q(\xi, \varphi) + q\left(\xi, \varphi + \frac{\pi}{2}\right) = A(\xi) + B(\xi)$$

we see that $A,\ B \in L^1(\Omega, m_0)$. For $\xi \in \Omega_0$ we introduce

$$G(\xi) = \begin{pmatrix} A(\xi) & C(\xi) \\ C(\xi) & B(\xi) \end{pmatrix},$$

whose positive eigenvalues $\lambda_1(\xi) \leq 1 \leq \lambda_2(\xi)$, satisfy $\lambda_1(\xi)\lambda_2(\xi) = 1$ and $\lambda_1(\xi) + \lambda_2(\xi) = A(\xi) + B(\xi)$. If $\mathbf{v}^t(\varphi) = (\cos\varphi, \sin\varphi)$ then

$$q(\xi, \varphi) = \mathbf{v}^t(\varphi) G(\xi) \mathbf{v}(\varphi).$$

In consequence

$$\lambda_1(\xi) \leq q(\xi, \varphi) \leq \lambda_2(\xi) \leq A(\xi) + B(\xi),$$

$$\lambda_1(\xi) \leq p(\xi, \varphi) \leq \lambda_2(\xi) \leq A(\xi) + B(\xi),$$

for every $\xi \in \Omega_0$. This shows that $p \in L^2(K_{\mathbb{R}}, m_1)$ and completes the proof of the statement. On the other hand

$$\int_{K_{\mathbb{R}}} p^2(\xi, \varphi)\, dm_1 \leq \int_\Omega \Big[\int_{P^1(\mathbb{R})} (A(\xi) + B(\xi)) p(\xi, \varphi) dr_1 \Big]\, dm_0,$$

hence

$$||p||_2^2 \leq \int_\Omega (A(\xi) + B(\xi))\, dm_0 = 2||q||_1. \tag{20}$$

We recall the linear invariant measure $d\mu = p_0 dm_1$, $p_0 = 1/q_0$, with q_0 given by (18). Notice that the above inequality (20) stablishes a direct relation between $||p_0||_2$ and $\alpha'(E)$. ∎

We are also in conditions to obtain quadratic on the fibers solutions of the functional equation when the Lyapunov exponent is positive.

PROPOSITION 3.6: *Let $E_0 \in \mathbb{R}$ with $\gamma(E_0) > 0$. If $\mathbf{v}^t(\varphi) = (\cos\varphi, \sin\varphi)$ and*

$$q(\xi, \varphi) = 2\mathbf{v}^t(\varphi) G(\xi, E_0) \mathbf{v}(\varphi); \tag{21}$$

then $\pm q$ are the unique quadratic on the fibers solutions of (12) whose coefficients satisfy $A(\xi)B(\xi) - C^2(\xi) = -1$ for almost every $\xi \in \Omega$.

Proof: Proposition 5.7 of [2] assures a unique quadratic on the fibers solution (except sign) of (12) with $A(\xi)B(\xi) - C^2(\xi) = -1$ for almost every $\xi \in \Omega$. We deduce from Theorem 3.3(ii) that it is associated to the matrix $2G(\xi, E_0)$. ∎

Formulas (18) and (21) permit the intérchange between ergodic and spectral relations which appear in the literature. The same kind of expressions can be obtained for the usual spectral problem defined by the second order differential equation.

Definition 3.7: We say that a map $P\colon \Omega \to SL(2,\mathbb{C})$ defines a **measurable Perron transformation** for m_0 if

(i) P is a measurable map.

(ii) There is an invariant subset $\Omega_0 \subset \Omega$ with $m_0(\Omega_0) = 1$ such that the map $P_\xi\colon \mathbb{R} \to \mathrm{SL}(2,\mathbb{C})$, $t \to P(\xi_t)$ is a C^1-map for every $\xi \in \Omega_0$.

Notice that $P^{-1}\colon \Omega \to \mathrm{SL}(2,\mathbb{C})$, $\xi \to P^{-1}(\xi)$ is also a measurable Perron transformation. We refer to P as an L^p-Perron transformation when the map $n\colon \Omega \to [0,\infty)$, $\xi \to ||P(\xi)|| + ||P^{-1}(\xi)||$ belongs to $L^p(\Omega, m_0)$. Obviously this definition is irrespective of the norm used on $\mathrm{SL}(2,\mathbb{C})$. We say that P is a strong Perron transformation if it is continuous and $\Omega_0 = \Omega$. Then P^{-1} is also a strong Perron transformation.

If the flow (Ω, Ξ) is distal and the solutions of (2) are bounded there exists a strong Perron transformation which takes the matrix S into a skew-symmetric one. In fact this is a general result for n-dimensional linear systems. (See [3], [11].) Every linear invariant measure gives rise to a measurable Perron transformation which preserves the measurable structure. If $(K_{\mathbb{R}}, \Phi)$ admits an absolutely continuous invariant measure there exists a measurable Perron transformation which takes the matrix S into a skew-symmetric one.

We can make this result more precise on $\mathbf{A}_2$. We take $E_0 \in \mathbf{A}_2$ and fix a linear invariant measure μ on $K_{\mathbb{R}}$ where $d\mu = p\,dm_1$ and

$$p(\xi,\varphi) = (A(\xi)\cos^2\varphi + B(\xi)\sin^2\varphi + 2C(\xi)\sin\varphi\cos\varphi)^{-1}.$$

It is known that the relations $B(\xi) = C^2(\xi)/A(\xi) + 1/A(\xi)$ hold for almost every $\xi \in \Omega$. Hence $C^2(\xi)/A(\xi)$, $1/A(\xi)$ belong to $L^1(\Omega, m_0)$. A direct application of Proposition 3.2 leads to

PROPOSITION 3.8: *Let us fix $E_0 \in \mathbf{A}_2$. The map*

$$\begin{aligned} P\colon \Omega &\longrightarrow \mathrm{SL}(2,\mathbb{C}) \\ \xi &\longrightarrow \begin{pmatrix} \dfrac{1}{\sqrt{A(\xi)}} & 0 \\ \dfrac{C(\xi)}{\sqrt{A(\xi)}} & \sqrt{A(\xi)} \end{pmatrix} \end{aligned} \tag{22}$$

is an L^2-Perron transformation. Moreover

(i) *The change of variables $\mathbf{w} = P(\xi_t)\mathbf{z}$ transforms the systems $\mathbf{z}' = S_{E_0}(\xi_t)\mathbf{z}$ into*

$$\mathbf{w}' = \begin{pmatrix} 0 & \dfrac{b(\xi_t) + E_0}{A(\xi_t)} \\ \dfrac{-b(\xi_t) - E_0}{A(\xi_t)} & 0 \end{pmatrix} \mathbf{w} \tag{23}$$

for almost every $\xi \in \Omega$.

(ii) *The Möbius–Perron transformation $W = A(\xi)Z + C(\xi)$ takes the Riccati equations $Z' = (-E_0 + c(\xi_t)) - 2a(\xi_t)Z - (E_0 + b(\xi_t))Z^2$ into*

$$W' = \frac{-b(\xi_t) - E_0}{A(\xi_t)}(1 + W^2)$$

for almost every $\xi \in \Omega$.

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